

CANONICAL DUALITY APPROACH IN THE APPROXIMATION OF OPTIMAL MONGE MASS TRANSFER MAPPING

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ABSTRACT. This paper mainly addresses the Monge mass transfer problem in the 1-D case. Through an ingenious approximation mechanism, one transforms the Monge problem into a sequence of minimization problems, which can be converted into a sequence of nonlinear differential equations with constraints by variational method. The existence and uniqueness of the solution for each equation can be demonstrated by applying the canonical duality method. Moreover, the duality method gives a sequence of perfect dual maximization problems. In the final analysis, one constructs the approximation of optimal mapping for the Monge problem according to the theoretical results.

RÉSUMÉ. Dans cet article, on considère essentiellement le problème des transports des masses de Monge en dimension 1. En utilisant un mécanisme d'approximation ingénieuse, on transforme le problème de Monge en une séquence de problèmes de minimisation, qui peut être convertie en une séquence des équations différentielles nonlinéaires avec des contraintes par la méthode variationnelle. En particulier, nous prouvons l'existence et l'unicité de la solution en appliquant la méthode de dualité canonique. De plus, la transformation de dualité donne une séquence des duals parfaits de maximisation. Enfin, nous étudions le problème de Monge selon les résultats théoriques.

1. INTRODUCTION

The Monge mass transfer model is widely used in modern social and economic activities, medical science and mechanical processes, etc. In these respects, some typical examples include the migration problem, distribution of industrial products, purification of blood in the kidneys and livers, shape optimization, etc. Interested readers can refer to [1, 2, 7, 22, 23, 28, 31] for more details.

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The original transfer problem, which was proposed by Monge [28], investigated how to move one mass distribution to another one with the least amount of work. In this paper, we consider the Monge problem in the 1-D case. Let $\Omega = [a, b]$ and $\Omega^* = [c, d]$, $a, b, c, d \in \mathbb{R}$. Here we focus on the closed case, and other bounded cases can be discussed similarly. Moreover, f^+ and f^- are two nonnegative density functions in Ω and Ω^* , respectively, and satisfy the normalized balance condition

$$\int_{\Omega} f^+ dx = \int_{\Omega^*} f^- dx = 1.$$

Let $c : \Omega \times \Omega^* \rightarrow [0, +\infty)$ be a cost function, which indicates the work required to move a unit mass from the position x to a new position y . There are many types of cost functions while dealing with different problems [1, 4, 7, 29]. In the Monge problem, the cost function is proportional to the distance $|x - y|$, for simplicity, $c(x, y) = |x - y|$. The Monge problem consists in finding an optimal mass transfer mapping $\mathbf{s}^* : \Omega \rightarrow \Omega^*$ to minimize the cost functional $I(\mathbf{s})$:

$$(1) \quad I(\mathbf{s}^*) = \min_{\mathbf{s} \in \mathcal{N}} \left\{ I[\mathbf{s}] := \int_{\Omega} |x - \mathbf{s}(x)| f^+(x) dx \right\},$$

where $\mathbf{s} : \Omega \rightarrow \Omega^*$ belongs to the class \mathcal{N} of measurable one-to-one mappings driving $f^+(x)$ to $f^-(y)$.

In the 1940s, Kantorovich initiated a duality theory by relaxing Monge transfer problem to the task of finding a maximizer for the Kantorovich problem [22, 23]. This mechanism plays an archetypal role in the infinite-dimensional linear programming [33]. As a matter of fact, the Kantorovich problem may not be a perfect dual to the Monge problem unless a so-called *dual criteria for optimality* is satisfied [4, 7]. Indeed, a huge body of mathematical tools have been developed for computing the maximizer, such as the Monge-Kantorovich-Rubinstein-Wasserstein matrices [30], dual potentials for capacity constrained optimal transport [21], etc.

It turns out that the nonuniform convexity of the cost function $c(x, y)$ defeats many simple attempts to sort out the structure of optimal mass allocation. In order to gain some insight into this problem, many mathematicians introduced lots of approximating mechanisms. For example, L. A. Caffarelli, W. Gangbo, R. J. McCann and X. J. Wang [5, 19, 20, 32], etc. utilized an approximation of strictly convex cost functions

$$c_{\epsilon}(x, y) = |x - y|^{1+\epsilon} \quad \epsilon > 0.$$

The existence and uniqueness of the optimal mapping \mathbf{s}_{ϵ}^* can be proved by convex analysis. Then let ϵ tends to 0, and one can construct an optimal mapping \mathbf{s}^* by using transfer rays and transfer sets invoked by L. C. Evans and W. Gangbo [8]. In addition, N. S. Trudinger and X. J. Wang etc. used the approximation

$$c_{\epsilon}(x, y) = \sqrt{\epsilon^2 + |x - y|^2}$$

in the discussion of regularity [29, 32]. Moreover, L. C. Evans, W. Gangbo and J. Moser [6, 7, 8] provided an ODE recipe to build \mathbf{s}^* by solving a flow problem involving

Du . This method is very useful but really complicated.

In this paper, we consider the approximation of an optimal mapping through solving the optimization of distribution density in the probability theory. Here, we mainly consider two typical cases of $\Omega \cap \Omega^* = \emptyset$, namely,

- **Assumption I:** $0 \leq c < d < a < b$, $a \gg 0$, $d - c$ sufficiently large, specified in Lemma 2.9;
- **Assumption II:** $a < b < c < d \leq 0$, $b \ll 0$, $d - c$ sufficiently large, specified in Lemma 2.9.

Let $\alpha > 0$ be sufficiently large and consider the distribution densities subject to

$$(2) \quad u \in W_0^{1,\infty}(\Omega^*) \cap C(\overline{\Omega^*}),$$

$$(3) \quad u \geq 0, \text{ a.e. in } \Omega^*,$$

$$(4) \quad \|u\|_{L^1(\Omega^*)} = 1,$$

$$(5) \quad \|u_y\|_{L^\infty(\Omega^*)} \leq \alpha,$$

where $W_0^{1,\infty}(\Omega^*)$ is the Sobolev space and u_y stands for the weak derivative with respect to $y \in \Omega^*$. It is evident that the δ -function is excluded from our discussion. On the one hand, under the Assumption I, Monge transfer problem (1) can be converted into a maximization of the expectation of the real-valued random variable $Y \in \Omega^*$ with respect to the distribution densities u subject to (2)-(5),

$$(6) \quad (\mathcal{P}^{(1)}) : \max_u \left\{ \mathbb{E}_u(Y) := \int_{\Omega^*} yu(y)dy \right\},$$

with

- **Assumption III:** The optimal mapping \mathbf{s}^* is strictly monotonous. Under Assumption I, $\mathbf{s}^*(a) = d$ or $\mathbf{s}^*(b) = d$; while under Assumption II, $\mathbf{s}^*(a) = c$ or $\mathbf{s}^*(b) = c$.

On the other hand, under the Assumptions II and III, Monge transfer problem (1) can be converted into a minimization of the expectation of the real-valued random variable $Y \in \Omega^*$ with respect to the distribution densities u subject to (2)-(5),,

$$(7) \quad (\mathcal{P}^{(2)}) : \min_u \left\{ \mathbb{E}_u(Y) := \int_{\Omega^*} yu(y)dy \right\}.$$

In this paper, we investigate the analytic approximating mapping through *canonical duality method* introduced by David. Y. Gao and G. Strang [12, 13, 14]. This theory was originally proposed to find minimizers for a non-convex strain energy functional with a double-well potential. During the last few years, considerable effort has been taken to illustrate these non-convex problems from the theoretical point of view. Through applying this method, David Y. Gao and G. Strang characterized the local energy extrema and the global energy minimizer for both hard device and soft device and finally obtained the analytical solutions. Readers can refer to [15, 16, 17, 18].

Inspired by the survey paper [9], we propose a nonlinear differential equation approach by introducing a sequence of approximation problems of the primal $(\mathcal{P}^{(1)})$ and $(\mathcal{P}^{(2)})$, namely,

$$(8) \quad (\mathcal{P}^{(\varepsilon)}) : \min_{w_\varepsilon} \left\{ I^{(\varepsilon)}[w_\varepsilon] := \int_{\Omega^*} L^{(\varepsilon)}(w_{\varepsilon,y}, w_\varepsilon, y) dy := \int_{\Omega^*} \left(H^{(\varepsilon)}(w_{\varepsilon,y}) - w_\varepsilon |y| \right) dy \right\},$$

where $H^{(\varepsilon)} : \mathbb{R} \rightarrow \mathbb{R}^+$ is defined as

$$H^{(\varepsilon)}(\gamma) := \varepsilon e^{(\gamma^2 - \alpha^2)/(2\varepsilon)}.$$

Moreover,

$$L^{(\varepsilon)}(P, z, y) : \mathbb{R} \times \mathbb{R} \times \Omega^* \rightarrow \mathbb{R}$$

satisfies the following coercivity inequality and is convex in the variable P ,

$$(9) \quad L^{(\varepsilon)}(P, z, y) \geq p_\varepsilon |P|^2 - q_\varepsilon, \quad P, z \in \mathbb{R}, y \in \Omega^*,$$

for constants p_ε and q_ε . $I^{(\varepsilon)}$ is called the potential energy functional and is weakly lower semicontinuous on $W_0^{1,\infty}(\Omega^*)$. It's worth noticing that when $|\gamma| \leq \alpha$, then

$$\lim_{\varepsilon \rightarrow 0^+} H^{(\varepsilon)}(\gamma) = 0$$

uniformly. From [10], one knows immediately there exists a distribution density \bar{u}_ε solving

$$I^{(\varepsilon)}[\bar{u}_\varepsilon] = \min_{w_\varepsilon} \left\{ I^{(\varepsilon)}[w_\varepsilon] \right\}.$$

Consequently, once such a sequence of functions $\{\bar{u}_\varepsilon\}_\varepsilon$ is obtained, then it will help find an optimal distribution density which solves the primal problems $(\mathcal{P}^{(1)})$ or $(\mathcal{P}^{(2)})$. This paper is aimed to obtain an explicit representation of this approximation sequence. Generally speaking, there are plenty of approximating schemes, for example, one can also let

$$H^{(\varepsilon)}(\gamma) := \varepsilon(\gamma^2 - \alpha^2)^2.$$

Then by following the procedure in dealing with double-well potentials in [12, 18], we could definitely find an optimal distribution density.

By variational calculus, correspondingly, one derives a sequence of Euler-Lagrange equations for $(\mathcal{P}^{(\varepsilon)})$,

$$(10) \quad (e^{(u_{\varepsilon,y}^2 - \alpha^2)/(2\varepsilon)} u_{\varepsilon,y})_y + |y| = 0, \quad \text{in } U^{(\varepsilon)},$$

equipped with the Dirichlet boundary condition, where the compact support

$$U^{(\varepsilon)} := \text{Supp}(u_\varepsilon) \subset \Omega^*$$

is connected and will be determined later. The term $e^{(u_{\varepsilon,y}^2 - \alpha^2)/(2\varepsilon)}$ is called the transport density. As a matter of fact, $\{u_\varepsilon\}_\varepsilon$ is a sequence of strictly concave functions. Clearly, like p -Laplacian, $e^{(u_{\varepsilon,y}^2 - \alpha^2)/(2\varepsilon)}$ is a highly nonlinear function, which is difficult to solve by the direct approach [3, 10, 24]. However, by the canonical duality theory, one is able to demonstrate the existence and uniqueness of the solution of the Euler-Lagrange equation, which establishes the equivalence between the local minimizer of $(\mathcal{P}^{(\varepsilon)})$ and the solution of Euler-Lagrange equation (5). This will help find a global minimizer of $(\mathcal{P}^{(\varepsilon)})$.

At the moment, we would like to introduce the main theorems. First, we consider the approximation problem of (8).

Theorem 1.1. *For any $\varepsilon > 0$, there exists a sequence of solutions $\{\bar{u}_\varepsilon\}_\varepsilon$ satisfying (2)-(5) for the Euler-Lagrange equations (10), which is at the same time a sequence of global minimizers for the approximation problems $(\mathcal{P}^{(\varepsilon)})$ in the following form,*

- Under Assumption I,

$$\bar{u}_\varepsilon(y) = \begin{cases} \int_d^y (-G(t) + C_\varepsilon(d))/E_\varepsilon^{-1}((-G(t) + C_\varepsilon(d))^2)dt, & y \in [p_\varepsilon^*(d), d] \subset \Omega^*, \\ 0, & \text{elsewhere in } \Omega^*; \end{cases}$$

- Under Assumption II,

$$\bar{u}_\varepsilon(y) = \begin{cases} \int_c^y (G(t) - D_\varepsilon(c))/E_\varepsilon^{-1}((G(t) - D_\varepsilon(c))^2)dt, & y \in [c, q_\varepsilon^*(c)] \subset \Omega^*, \\ 0, & \text{elsewhere in } \Omega^*, \end{cases}$$

where E_ε and G are defined as

$$\begin{cases} E_\varepsilon(\gamma) := \gamma^2 \ln(e^{\alpha^2} \gamma^{2\varepsilon}), & \gamma \in [e^{-\alpha^2/(2\varepsilon)}, 1], \\ G(y) := y^2/2, & y \in [p_\varepsilon^*(d), d] \text{ or } [c, q_\varepsilon^*(c)]. \end{cases}$$

E_ε^{-1} stands for the inverse of E_ε , $C_\varepsilon(d)$ and $p_\varepsilon^*(d)$ are constants depending on d and ε , while $D_\varepsilon(c)$ and $q_\varepsilon^*(c)$ are constants depending on c and ε .

By letting $\varepsilon \rightarrow 0^+$, one can solve the optimization problems for the expectation of the real-valued variable $Y \in \Omega^*$.

Theorem 1.2. *For the maximization problem $(\mathcal{P}^{(1)})$ (or the minimization problem $(\mathcal{P}^{(2)})$), there exists a global maximizing (or minimizing) distribution density f^- satisfying (2)-(5).*

Furthermore, under Assumption III, one is able to deal with the Monge transfer problem (1) with the optimal distribution densities. In the following, we construct a sequence of mass transfer mappings \mathbf{s}_ε approximating an optimal mapping \mathbf{s}^* . Let

$$F(x) := \int_a^x f^+(t)dt, \quad x \in [a, b].$$

If $f^+ > 0$, then F is monotonously increasing with respect to $x \in [a, b]$, invertible and its inverse is denoted as

$$F^{-1} : [0, 1] \rightarrow [a, b].$$

In addition, let

$$Q_\varepsilon(y) := \int_{p_\varepsilon^*(d)}^y \bar{u}_\varepsilon(t)dt, \quad y \in [p_\varepsilon^*(d), d].$$

Since $\bar{u}_\varepsilon > 0$ in $(p_\varepsilon^*(d), d)$, then Q_ε is monotonously increasing with respect to $y \in [p_\varepsilon^*(d), d]$, invertible and its inverse is denoted as

$$Q_\varepsilon^{-1} : [0, 1] \rightarrow [p_\varepsilon^*(d), d].$$

Furthermore, let

$$R_\varepsilon(y) := \int_c^y \bar{u}_\varepsilon(t) dt, \quad y \in [c, q_\varepsilon^*(c)].$$

Since $\bar{u}_\varepsilon > 0$ in $(c, q_\varepsilon^*(c))$, then R_ε is monotonously increasing with respect to $y \in [c, q_\varepsilon^*(c)]$, invertible and its inverse is denoted as

$$R_\varepsilon^{-1} : [0, 1] \rightarrow [c, q_\varepsilon^*(c)].$$

Theorem 1.3. *Assume that $f^+(x) > 0$, $x \in [a, b]$. For the Monge transfer problem (1) under Assumption I and Assumption III, there exists a sequence of strictly increasing (or decreasing) mappings represented explicitly as*

$$\mathbf{s}_\varepsilon(x) = Q_\varepsilon^{-1}(F(x)), \quad x \in [a, b];$$

or

$$\mathbf{s}_\varepsilon(x) = Q_\varepsilon^{-1}(1 - F(x)), \quad x \in [a, b].$$

While for the Monge transfer problem (1) under Assumption II and Assumption III, there exists a sequence of strictly increasing (or decreasing) mappings represented explicitly as

$$\mathbf{s}_\varepsilon(x) = R_\varepsilon^{-1}(F(x)), \quad x \in [a, b];$$

or

$$\mathbf{s}_\varepsilon(x) = R_\varepsilon^{-1}(1 - F(x)), \quad x \in [a, b].$$

The rest of the paper is organized as follows. In Section 2, first we introduce some useful notations which will simplify the proof considerably. Then we apply the canonical dual transformation to deduce a sequence of perfect dual problems $(\mathcal{P}_d^{(\varepsilon)})$, corresponding to $(\mathcal{P}^{(\varepsilon)})$ and a pure complementary energy principle. Next we apply the canonical duality theory to prove Theorem 1.1, 1.2 and 1.3. A few remarks will conclude our discussion.

2. PROOF OF THE MAIN RESULTS

2.1. Useful notations. Before proving the main results, first and foremost, we introduce some useful notations.

- θ_ε is the corresponding Gâteaux derivative of $H^{(\varepsilon)}$ with respect to $u_{\varepsilon, y}$ given by

$$\theta_\varepsilon(y) = e^{(u_{\varepsilon, y}^2 - \alpha^2)/(2\varepsilon)} u_{\varepsilon, y}.$$

- $\Phi^{(\varepsilon)}$ is a nonlinear geometric mapping given by

$$\Phi^{(\varepsilon)}(u_\varepsilon) := (u_{\varepsilon, y}^2 - \alpha^2)/(2\varepsilon).$$

For convenience's sake, denote

$$\xi_\varepsilon := \Phi^{(\varepsilon)}(u_\varepsilon).$$

It is evident that ξ_ε belongs to the function space \mathcal{U} given by

$$\mathcal{U} := \left\{ \phi \mid \phi \leq 0 \right\}.$$

- $\Psi^{(\varepsilon)}$ is a canonical energy defined as

$$\Psi^{(\varepsilon)}(\xi_\varepsilon) := \varepsilon e^{\xi_\varepsilon},$$

which is a convex function with respect to ξ_ε .

- ζ_ε is the corresponding Gâteaux derivative of $\Psi^{(\varepsilon)}$ with respect to ξ_ε given by

$$\zeta_\varepsilon = \varepsilon e^{\xi_\varepsilon},$$

which is invertible with respect to ξ_ε and belongs to the function space $\mathcal{V}^{(\varepsilon)}$,

$$\mathcal{V}^{(\varepsilon)} := \left\{ \phi \mid 0 < \phi \leq \varepsilon \right\}.$$

- $\Psi_*^{(\varepsilon)}$ is defined as

$$\Psi_*^{(\varepsilon)}(\zeta_\varepsilon) := \xi_\varepsilon \zeta_\varepsilon - \Psi^{(\varepsilon)}(\xi_\varepsilon) = \zeta_\varepsilon (\ln(\zeta_\varepsilon/\varepsilon) - 1).$$

- λ_ε is defined as

$$\lambda_\varepsilon := \zeta_\varepsilon/\varepsilon,$$

and belongs to the function space \mathcal{V} ,

$$\mathcal{V} := \left\{ \phi \mid 0 < \phi \leq 1 \right\}.$$

2.2. Canonical duality techniques.

Definition 2.1. By Legendre transformation, one defines a Gao-Strang total complementary energy functional $\Xi^{(\varepsilon)}$,

$$\Xi^{(\varepsilon)}(u_\varepsilon, \zeta_\varepsilon) := \int_{U^{(\varepsilon)}} \left\{ \Phi^{(\varepsilon)}(u_\varepsilon) \zeta_\varepsilon - \Psi_*^{(\varepsilon)}(\zeta_\varepsilon) - |y| u_\varepsilon \right\} dy.$$

Next we introduce an important *criticality criterium* for $\Xi^{(\varepsilon)}$.

Definition 2.2. $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$ is called a critical pair of $\Xi^{(\varepsilon)}$ if and only if

$$(11) \quad D_{u_\varepsilon} \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = 0,$$

and

$$(12) \quad D_{\zeta_\varepsilon} \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = 0,$$

where $D_{u_\varepsilon}, D_{\zeta_\varepsilon}$ denote the partial Gâteaux derivatives of $\Xi^{(\varepsilon)}$, respectively.

Indeed, by variational calculus, we have the following observation from (11) and (12).

Lemma 2.3. On the one hand, for any fixed $\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}$, (11) is equivalent to the equilibrium equation

$$(\lambda_\varepsilon \bar{u}_{\varepsilon, y})_y + |y| = 0, \quad \text{in } U^{(\varepsilon)}.$$

On the other hand, for any fixed u_ε satisfying (2)-(5), (12) is consistent with the constructive law

$$\Phi^{(\varepsilon)}(u_\varepsilon) = D_{\zeta_\varepsilon} \Psi_*^{(\varepsilon)}(\bar{\zeta}_\varepsilon).$$

Lemma 2.3 indicates that \bar{u}_ε from the critical pair $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$ solves the Euler-Lagrange equation (10).

Definition 2.4. From Definition 2.1, one defines the Gao-Strang pure complementary energy $I_d^{(\varepsilon)}$ in the form

$$I_d^{(\varepsilon)}[\zeta_\varepsilon] := \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \zeta_\varepsilon),$$

where \bar{u}_ε solves the Euler-Lagrange equation (10).

For convenience's sake, we give another representation of the pure energy $I_d^{(\varepsilon)}$ by the following lemma.

Lemma 2.5. The pure complementary energy functional $I_d^{(\varepsilon)}$ can be rewritten as

$$I_d^{(\varepsilon)}[\zeta_\varepsilon] = -1/2 \int_{U^{(\varepsilon)}} \left\{ \varepsilon \theta_\varepsilon^2 / \zeta_\varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy,$$

where θ_ε satisfies

$$(13) \quad \theta_{\varepsilon,y} + |y| = 0 \text{ in } U^{(\varepsilon)},$$

equipped with a hidden boundary condition.

Proof. Through integrating by parts, one has

$$\begin{aligned} I_d^{(\varepsilon)}[\zeta_\varepsilon] &= - \underbrace{\int_{U^{(\varepsilon)}} \left\{ (\zeta_\varepsilon \bar{u}_{\varepsilon,y} / \varepsilon)_y + |y| \right\} \bar{u}_\varepsilon dy}_{(I)} \\ &\quad - 1/2 \underbrace{\int_{U^{(\varepsilon)}} \left\{ \zeta_\varepsilon \bar{u}_{\varepsilon,y}^2 / \varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy}_{(II)}. \end{aligned}$$

Since \bar{u}_ε solves the Euler-Lagrange equation (10), then the first part (I) disappears. Keeping in mind the definition of θ_ε and ζ_ε , one reaches the conclusion. \square

With the above discussion, next we establish a sequence of dual variational problems to the approximation problems $(\mathcal{P}^{(\varepsilon)})$.

$$(14) \quad (\mathcal{P}_d^{(\varepsilon)}) : \max_{\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}} \left\{ I_d^{(\varepsilon)}[\zeta_\varepsilon] = -1/2 \int_{U^{(\varepsilon)}} \left\{ \varepsilon \theta_\varepsilon^2 / \zeta_\varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy \right\}.$$

Indeed, by calculating the Gâteaux derivative of $I_d^{(\varepsilon)}$ with respect to ζ_ε , one has

Lemma 2.6. The variation of $I_d^{(\varepsilon)}$ with respect to ζ_ε leads to the dual algebraic equation (DAE), namely,

$$(15) \quad \theta_\varepsilon^2 = \bar{\zeta}_\varepsilon^2 (2 \ln(\bar{\zeta}_\varepsilon / \varepsilon) + \alpha^2 / \varepsilon) / \varepsilon,$$

where $\bar{\zeta}_\varepsilon$ is from the critical pair $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$.

Taking into account the notation of λ_ε , the identity (15) can be rewritten as

$$(16) \quad \theta_\varepsilon^2 = E_\varepsilon(\lambda_\varepsilon) = \lambda_\varepsilon^2 \ln(e^{\alpha^2} \lambda_\varepsilon^{2\varepsilon}).$$

It is evident E_ε is monotonously increasing with respect to $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$. As a matter of fact, θ_ε^2 has the following asymptotic expansion by using Taylor's expansion formula for $\ln \lambda_\varepsilon$ at the point 1.

Lemma 2.7. *When ε is sufficiently small, θ_ε^2 has the asymptotic expansion,*

$$\theta_\varepsilon^2 = (\alpha^2 - 2\varepsilon)\lambda_\varepsilon^2 + 2\varepsilon\lambda_\varepsilon^3 + R_\varepsilon(\lambda_\varepsilon),$$

where the remainder term

$$|R_\varepsilon(\lambda_\varepsilon)| \leq \varepsilon$$

uniformly for any $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$.

2.3. Proof of Theorem 1.1. From the above discussion, one deduces that, once θ_ε is given, then the analytic solution of the Euler-Lagrange equation (10) can be represented as

$$(17) \quad \bar{u}_\varepsilon(y) = \int_{y_0}^y \eta_\varepsilon(t) dt,$$

where $y \in U^{(\varepsilon)}$, $y_0 \in \partial U^{(\varepsilon)}$, $\eta_\varepsilon := \theta_\varepsilon/\lambda_\varepsilon$. In the following, we will determine the support $U^{(\varepsilon)}$. First and foremost, we prove several useful lemmas.

Lemma 2.8. *For $\forall \varepsilon > 0$,*

- *Under Assumption I, $\forall s \in [c, d]$, there exists a unique solution $\bar{u}_\varepsilon \in C^\infty[s, d]$ of the Euler-Lagrange equation (10) with Dirichlet boundary in the form of (17).*
- *Under Assumption II, for $\forall t \in (c, d]$, there exists a unique solution $\bar{u}_\varepsilon \in C^\infty[c, t]$ of the Euler-Lagrange equation (10) with Dirichlet boundary in the form of (17).*

Proof. First case:

In $[s, d]$, one has a general solution for the differential equation $\theta_{\varepsilon,y} = -y$ in the form of

$$\theta_\varepsilon(y) = -G(y) + C_\varepsilon = -y^2/2 + C_\varepsilon, \quad y \in [s, d] \subset (0, +\infty).$$

From the identity (16), one sees that there exists a unique C^∞ function $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$. By paying attention to the Dirichlet boundary $\bar{u}_\varepsilon(s) = 0$, one has the analytic solution \bar{u}_ε in the following form,

$$\bar{u}_\varepsilon(y) = \int_s^y \eta_\varepsilon(x) dx, \quad y \in [s, d].$$

Recall that

$$\bar{u}_\varepsilon(d) = \int_s^{G^{-1}(C_\varepsilon)} \eta_\varepsilon(x) dx + \int_{G^{-1}(C_\varepsilon)}^d \eta_\varepsilon(x) dx = 0,$$

and one can determine the constant $C_\varepsilon \in (s^2/2, d^2/2)$ uniquely. Indeed, let

$$\mu_\varepsilon(y, r) := (-G(y) + r)/\lambda_\varepsilon(y, r)$$

and

$$M_\varepsilon(r) := \int_s^{G^{-1}(r)} \mu_\varepsilon(y, r) dy + \int_{G^{-1}(r)}^d \mu_\varepsilon(y, r) dy,$$

where $\lambda_\varepsilon(y, r)$ is from (16). It is evident that λ_ε depends on C_ε . As a matter of fact, M_ε is strictly increasing with respect to $r \in (s^2/2, d^2/2)$, which leads to

$$C_\varepsilon = M_\varepsilon^{-1}(0).$$

In fact, C_ε depends on s and the contradiction method shows that C_ε is strictly increasing with respect to $s \in [c, d]$.

Second case:

In $[c, t]$, one has a general solution for the differential equation $\theta_{\varepsilon, y} = y$ in the form of

$$\theta_\varepsilon(y) = G(y) - D_\varepsilon = y^2/2 - D_\varepsilon, \quad y \in [c, t] \subset (-\infty, 0).$$

From the identity (16), one sees that there exists a unique C^∞ function $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$. By paying attention to the Dirichlet boundary $\bar{u}_\varepsilon(c) = 0$, one has the analytic solution \bar{u}_ε in the following form,

$$\bar{u}_\varepsilon(y) = \int_c^y \eta_\varepsilon(x) dx, \quad y \in [c, t].$$

Recall that

$$\bar{u}_\varepsilon(t) = \int_c^{G^{-1}(D_\varepsilon)} \eta_\varepsilon(x) dx + \int_{G^{-1}(D_\varepsilon)}^t \eta_\varepsilon(x) dx = 0,$$

and one can determine the constant $D_\varepsilon \in (t^2/2, c^2/2)$ uniquely. Indeed, let

$$\mu_\varepsilon(y, r) := (G(y) - r)/\lambda_\varepsilon(y, r)$$

and

$$N_\varepsilon(r) := \int_c^{G^{-1}(r)} \mu_\varepsilon(y, r) dy + \int_{G^{-1}(r)}^t \mu_\varepsilon(y, r) dy,$$

where $\lambda_\varepsilon(y, r)$ is from (16). It is evident that λ_ε depends on D_ε . As a matter of fact, N_ε is strictly decreasing with respect to $r \in (t^2/2, c^2/2)$, which leads to

$$D_\varepsilon = N_\varepsilon^{-1}(0).$$

In fact, D_ε depends on t and the contradiction method shows that D_ε is strictly decreasing with respect to $t \in (c, d]$.

□

Lemma 2.9. *For $\forall \varepsilon > 0$,*

- *Under Assumption I, if $d - c$ is sufficiently large such that*

$$(18) \quad \int_c^d \int_d^y \left(-x^2/2 + C_\varepsilon \right) / \left(\lambda_\varepsilon(x, C_\varepsilon) \right) dx dy > 1 \text{ when } d \in \text{Supp } \bar{u}_\varepsilon,$$

then there exists a unique $p_\varepsilon^*(d)$ such that

$$\text{Supp } \bar{u}_\varepsilon = [p_\varepsilon^*(d), d] \text{ and } \int_{p_\varepsilon^*(d)}^d \bar{u}_\varepsilon(y) dy = 1.$$

- Under Assumption II, if $d - c$ is sufficiently large such that

$$(19) \quad \int_c^d \int_c^y \left(x^2/2 - D_\varepsilon \right) / \left(\lambda_\varepsilon(x, D_\varepsilon) \right) dx dy > 1 \text{ when } c \in \text{Supp } \bar{u}_\varepsilon,$$

then there exists a unique $q_\varepsilon^*(c)$ such that

$$\text{Supp } \bar{u}_\varepsilon = [c, q_\varepsilon^*(c)] \text{ and } \int_c^{q_\varepsilon^*(c)} \bar{u}_\varepsilon(y) dy = 1.$$

Proof. First Part:

Let $\text{Supp } \bar{u}_\varepsilon = [s, d]$ and define a function $\Pi : [c, d] \rightarrow \mathbb{R}^+$ as follows,

$$\Pi(s) := \int_s^d \int_d^y \left(-x^2/2 + C_\varepsilon(s) \right) / \left(\lambda_\varepsilon(x, C_\varepsilon(s)) \right) dx dy.$$

Indeed, since C_ε is strictly increasing with respect to $s \in [c, d]$, as a result, it is easy to check that Π is a strictly decreasing function with respect to $s \in [c, d]$. The first assertion follows immediately when we recall (18).

Second Part:

Let $\text{Supp } \bar{u}_\varepsilon = [c, t]$ and define a function $\Pi : (c, d] \rightarrow \mathbb{R}^+$ as follows,

$$\Pi(t) := \int_c^t \int_c^y \left(x^2/2 - D_\varepsilon(t) \right) / \left(\lambda_\varepsilon(x, D_\varepsilon(t)) \right) dx dy.$$

Indeed, since D_ε is strictly decreasing with respect to $t \in [c, d]$, as a result, it is easy to check that Π is a strictly increasing function with respect to $t \in (c, d]$. The second assertion follows immediately when we recall (19). \square

Next we verify that \bar{u}_ε is exactly a global minimizer for $(\mathcal{P}^{(\varepsilon)})$ and $\bar{\zeta}_\varepsilon$ is a global maximizer for $(\mathcal{P}_d^{(\varepsilon)})$.

Lemma 2.10. (Canonical Duality Theory) For $\forall \varepsilon > 0$, \bar{u}_ε in Lemma 2.9 is a global minimizer for the approximation problem $(\mathcal{P}^{(\varepsilon)})$. And the corresponding $\bar{\zeta}_\varepsilon$ is a global maximizer for the dual problem $(\mathcal{P}_d^{(\varepsilon)})$. Moreover, the following duality identity holds,

$$(20) \quad I^{(\varepsilon)}[\bar{u}_\varepsilon] = \min_{u_\varepsilon} I^{(\varepsilon)}[u_\varepsilon] = \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = \max_{\zeta_\varepsilon} I_d^{(\varepsilon)}[\zeta_\varepsilon] = I_d^{(\varepsilon)}[\bar{\zeta}_\varepsilon],$$

where u_ε is subject to the constraints (2)-(5) and $\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}$.

Lemma 2.10 demonstrates that the maximization of the pure complementary energy functional $I_d^{(\varepsilon)}$ is perfectly dual to the minimization of the potential energy functional $I^{(\varepsilon)}$. In effect, the identity (20) indicates there is no duality gap between them.

Proof. On the one hand, for any function $\phi \in W_0^{1,\infty}(U^{(\varepsilon)})$, the second variational form $\delta_\phi^2 I^{(\varepsilon)}$ is equal to

$$(21) \quad \int_{U^{(\varepsilon)}} e^{(\bar{u}_{\varepsilon,x}^2 - \alpha^2)/(2\varepsilon)} \left\{ (\bar{u}_{\varepsilon,x} \phi_x)^2 / \varepsilon + \phi_x^2 \right\} dx.$$

On the other hand, for any function $\psi \in \mathcal{V}^{(\varepsilon)}$, the second variational form $\delta_\psi^2 I_d^{(\varepsilon)}$ is equal to

$$(22) \quad - \int_{U^{(\varepsilon)}} \left\{ \varepsilon \theta_\varepsilon^2 \psi^2 / \bar{\zeta}_\varepsilon^3 + \psi^2 / \bar{\zeta}_\varepsilon \right\} dx.$$

From (21) and (22), one deduces immediately that

$$\delta_\phi^2 I^{(\varepsilon)}[\bar{u}_\varepsilon] \geq 0, \quad \delta_\psi^2 I_d^{(\varepsilon)}[\bar{\zeta}_\varepsilon] \leq 0.$$

□

Consequently, we reach the conclusion of Theorem 1.1 by summarizing the above discussion.

2.4. Proof of Theorem 1.2. According to Rellich-Kondrachov Compactness Theorem, since

$$\sup_\varepsilon |\bar{u}_\varepsilon| \leq \alpha(d - c)$$

and

$$\sup_\varepsilon |\bar{u}_{\varepsilon,y}| \leq \alpha,$$

then, there exists a subsequence $\{\bar{u}_{\varepsilon_k}\}_{\varepsilon_k}$ and $f^- \in W_0^{1,\infty}(\Omega^*) \cap C(\overline{\Omega^*})$ such that

$$(23) \quad \bar{u}_{\varepsilon_k} \rightarrow f^- \quad (k \rightarrow \infty) \text{ in } L^\infty(\Omega^*),$$

$$(24) \quad \bar{u}_{\varepsilon_k,y} \rightharpoonup_* f_y^- \quad (k \rightarrow \infty) \text{ weakly } * \text{ in } L^\infty(\Omega^*).$$

It remains to check that f^- satisfies (2)-(5). From (23), one knows

$$(25) \quad \bar{u}_{\varepsilon_k} \rightarrow f^- \quad (k \rightarrow \infty) \text{ a.e. in } \Omega^*.$$

According to Lebesgue's dominated convergence theorem,

$$\int_{\Omega^*} f^-(y) dy = \lim_{k \rightarrow \infty} \int_{\Omega^*} \bar{u}_{\varepsilon_k}(y) dy = 1.$$

From (24), one has

$$\|f_y^-\|_{L^\infty(\Omega^*)} \leq \liminf_{k \rightarrow \infty} \|\bar{u}_{\varepsilon_k,y}\|_{L^\infty(\Omega^*)} \leq \sup_{k \rightarrow \infty} \|\bar{u}_{\varepsilon_k,y}\|_{L^\infty(\Omega^*)} \leq \alpha.$$

2.5. Proof of Theorem 1.3. On the one hand, under Assumption I and III, one solves the differential problems

$$\bar{u}_\varepsilon(\mathbf{s}_\varepsilon)d\mathbf{s}_\varepsilon = f^+(x)dx, \quad \mathbf{s}_\varepsilon(b) = d,$$

or

$$\bar{u}_\varepsilon(\mathbf{s}_\varepsilon)d\mathbf{s}_\varepsilon = -f^+(x)dx, \quad \mathbf{s}_\varepsilon(a) = d$$

respectively. On the other hand, under Assumption II and III, one solves the differential problems

$$\bar{u}_\varepsilon(\mathbf{s}_\varepsilon)d\mathbf{s}_\varepsilon = f^+(x)dx, \quad \mathbf{s}_\varepsilon(a) = c,$$

or

$$\bar{u}_\varepsilon(\mathbf{s}_\varepsilon)d\mathbf{s}_\varepsilon = -f^+(x)dx, \quad \mathbf{s}_\varepsilon(b) = c$$

respectively. Our conclusion follows immediately.

Concluding Remarks:

In this paper, we mainly focus on the construction of 1-1 mappings approximating the optimal Monge transfer mapping. Rather than numerical simulation, we give the explicit representation of the approximating mappings by applying the canonical duality method. Together with the other convex approximation mechanisms used by L. A. Caffarelli, W. Gangbo, R. J. McCann, N. S. Trudinger, L. C. Evans, W. Gangbo and X. J. Wang [5, 6, 7, 8, 19, 20, 29, 32], we provide another viewpoint, namely, nonlinear differential equation approach, for the Monge transfer problem.

As a matter of fact, in a similar manner, Assumption I and Assumption II can be relaxed to the whole \mathbb{R} as long as $\Omega \cap \Omega^* = \emptyset$. It remains to discuss various cases when $\Omega \cap \Omega^* \neq \emptyset$, in which case, several new assumptions will have to be introduced. Furthermore, optimal transfer mapping in the n -dimensional case will be given in a sequential paper. The canonical duality method proves to be useful and can also be applied in the discussion of p -Laplacian problems and optimal probability density for p -th moment etc. [26, 27].

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REFERENCES

- [1] L. Ambrosio, Optimal transport maps in Monge-Kantorovich problem. ICM, **3**(2002), 1-3.
- [2] L. Ambrosio, Lecture Notes on Optimal Transfer Problems. preprint
- [3] Bourgain J., Brezis, H.: Sur l'équation $\operatorname{div} u = f$. C. R. Acad. Sci. Paris, Ser. I **334**(2002), 973-976.
- [4] L. A. Caffarelli, M. Feldman, R. J. McCann, Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs. Journal of AMS, **15**(2001), 1-26.
- [5] L. A. Caffarelli, Allocation maps with general cost functions, in Partial Differential Equations and Applications. Lecture Notes in Pure and Appl. Math. **177**(1996), 29-35.
- [6] B. Dacorogna and J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. H. Poincaré, **7**(1990), 1-26.
- [7] L. C. Evans, Partial differential equations and Monge-Kantorovich mass transfer(survey paper).
- [8] L. C. Evans, W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem. Mem. Amer. Math. Soc. **653**, 1999.
- [9] L. C. Evans, Three singular variational problems. preprint, 2002.
- [10] L. C. Evans, Partial Differential Equations. Graduate Studies in Mathematics, Vol. **19**, 2002.
- [11] W. Gangbo, A. Świąch, Optimal maps for the multidimensional Monge-Kantorovich problem. preprint, 1996.
- [12] D. Y. Gao, G. Strang, Geometric nonlinearity: Potential energy, complementary energy, and the gap function. Quart. Appl. Math. **47**(3)(1989), 487-504.
- [13] D. Y. Gao, Duality, triality and complementary extremum principles in non-convex parametric variational problems with applications. IMA Journal of Applied Mathematics **61**(1998), 199-235.
- [14] D. Y. Gao, Duality Principles in Nonconvex Systems: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
- [15] D. Y. Gao, Analytic solution and triality theory for nonconvex and nonsmooth variational problems with applications. Nonlinear Analysis, **42**(7)(2000), 1161-1193.
- [16] D. Y. Gao, R. W. Ogden, G. Stravroulakis, Nonsmooth and Nonconvex Mechanics: Modelling, Analysis and Numerical Methods. Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
- [17] D. Y. Gao, R. W. Ogden, Multiple solutions to non-convex variational problems with implications for phase transitions and numerical computation. Q. Jl Mech. Appl. Math. **61**(4)(2008), 497-522.
- [18] D. Y. Gao, X. Lu, Multiple solutions for non-convex variational boundary value problems in higher dimensions, preprint, 2013.
- [19] W. Gangbo, R. J. McCann, Optimal maps in Monge's mass transport problem. C. R. Acad. Sci. Paris Sér. I Math., **321**(1995), 1653-1658.
- [20] W. Gangbo, R. J. McCann, The geometry of optimal transportation. Acta Math., **177**(1996), 113-161.
- [21] J. Korman, R. J. McCann, C. Seis, Dual potentials for capacity constrained optimal transport. Calc. Var., **54**(2015), 573-584.
- [22] L. V. Kantorovich, On the transfer of masses. Dokl. Akad. Nauk. SSSR **37**(1942), 227-229(Russian).
- [23] L. V. Kantorovich, On a problem of Monge. Uspekhi Mat. Nauk. **3**(1948), 225-226.
- [24] J. L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications I-III, Dunod, Paris, 1968-1970.
- [25] Q. R. Li, R. Santambrogio, X. J. Wang, Regularity in Monge's mass transfer problem. J. Math. Pures Appl. **102**(2014), 1015-1040.
- [26] X. Lu and Y. Wu, Explicit representation for p-Laplacian problem. preprint, 2015.
- [27] Y. Wu and X. Lu, An approximation method for the optimization of p -th moment of \mathbb{R}^n -valued random variable. preprint, 2015.
- [28] G. Monge, Mémoire sur la théorie des déblais et de remblais. Histoire de l'Académie Royale des Sciences de Paris, avec les Mémoires de Mathématique et de Physique pour la même année, (1781), 666-704.

- [29] X. N. Ma, N. S. Trudinger, X. J. Wang, Regularity of potential functions of the optimal transportation problem. *Arch. Rational Mech. Anal.* **177**(2005), 151-183.
- [30] S. T. Rachev, The Monge-Kantorovich mass transference problem and its stochastic applications. *Theory of Prob. and Appl.*, **29**(1984), 647-676.
- [31] V. N. Sudakov, Geometric problems in the theory of infinite-dimensional probability distributions. *Proceedings of Steklov Institute* **141**(1979), 1-178.
- [32] N. S. Trudinger, X. J. Wang, On the Monge mass transfer problem. *Calc. Var.* **13**(2001), 19-31.
- [33] A. M. Vershik, Some remarks on the infinite-dimensional problems of linear programming, *Russian Math. Survey*, **25**(1970), 117-124.